Applied Mathematical Sciences

Volume 191

More information about this series at http://www.springer.com/series/34

Christian Kuehn

Multiple Time Scale Dynamics

Springer

Christian Kuehn
Institute for Analysis
and Scientific Computing
Vienna University of Technology
Vienna, Austria

ISSN 0066-5452 ISSN 2196-968X (electronic)
ISBN 978-3-319-12315-8 ISBN 978-3-319-12316-5 (eBook)
DOI 10.1007/978-3-319-12316-5
Springer Cham Heidelberg New York Dordrecht London

Library of Congress Control Number: 2014955212

Mathematics Subject Classification: 34-01, 34-02, 34A26, 34A34, 34C26, 34C45, 37-01, 37-02, 34Exx, 37Gxx

Printed on acid-free paper

Springer is part of Springer Science+Business Media (www.springer.com)

Preface

This book aims to provide an introduction to dynamical systems with multiple time scales. As in any overview book, several topics are covered only quite briefly. My aim was to focus on topics that seem to be less available in introductory form. However, I try to give a global view of the subject by covering a broad spectrum of ideas and tools. The detailed bibliography aims to direct the reader to further topics. To explain it with a simple metaphor: using this book should make you more familiar with a country's map, culture, and main attractions rather than imparting details of every street in just one city. Both things are useful at times.

The term "multiple time scale dynamics" is rather modern. The subject and many of its core ideas are much older. For example, "singular perturbation theory" or "multiscale systems" encompass a larger variety of topics than what I present here. On the one hand, no serious multidimensional spatial problems are considered in this book. Furthermore, there are many singularly perturbed problems that have very little to do with dynamical systems. On the other hand, ordinary differential equations (ODEs) with multiple time scales already contain motivation, technique, and intuition for more complicated scenarios.

Classical singular perturbation theory for multiple time scale systems provides many asymptotic techniques centered on series expansions, matching, and averaging. These methods are still indispensable today, and this book gives an overview of them. However, the details are not covered, since many excellent introductory texts are available. The last two decades have brought major additional progress with a particular focus on geometric ideas as well as powerful numerical algorithms. A major goal of this book was to merge several viewpoints with a wide variety of different techniques into a unified framework. Another reason for the broad choice of topics was to make it easier for students and researchers new to the field to get a much quicker overview.

Again, I would like to warn the reader that this book is obviously not a mathematical monograph aiming at a complete treatment of the entire field of multiple time scales. Some readers, particularly students, may wonder how a book of over 700 pages can be only an "introduction," but let me point out that most chapters, and even many five-page sections, in this book in fact deserve their own mathematical monograph of 300 pages or more. A few such books have been written, while many exist only in a distant, happier future. I encourage my colleagues working in the field—you know who you are—to begin work on such projects and fill in the missing mathematical details that I decided to leave out in order to make the subject much more accessible to beginners. Despite the simplifications, there seem to be several advantages of the style of presentation. The great diversity of the subject, ranging from mathematical theory in dynamics, analysis, geometry, topology, stochastics, and numerics to virtually all fields in science and engineering applications, easily becomes visible. The unity and interconnections between different approaches to multiple time scale problems can be identified much more readily. Also, scientists with particular applications in mind should find it easier to spot many potential tools right away, while a "purer" mathematician can use this text as a source book of open mathematical problems. The target audience of the book is senior undergraduates, graduate students, as well as researchers interested in using the theory of multiple time scale dynamics in nonlinear science, either from a theoretical or a mathematical modeling perspective. Section 1.1 provides a more detailed guide to the book.

Now I have the pleasure of thanking several colleagues, collaborators, and institutions that have helped to get this book started, keep it on track, and eventually push it over the finish line. First and foremost, I would like to thank my thesis adviser, John Guckenheimer, for introducing me to the field during my time as a graduate student. Undoubtedly, he shaped my view of the field, and without his support and encouragement, I would never have attempted to undertake a book project on multiple time scale systems. Important influences on this book during my postdoctoral years have come from my colleagues Thilo Gross, Peter Szmolyan, Nils Berglund, and Barbara Gentz. Thilo helped me to form bridges from multiscale dynamics to such seemingly distant areas as ecology, networks, systems biology, and statistical physics. I would like to thank Peter for sharing his tremendous insights into all aspects of geometric multiscale dynamics. Nils and Barbara have been constant sources of inspiration on everything stochastic. Although it is clear that I am responsible for all potential errors that may remain within this version, I would like to thank several colleagues who responded with valuable feedback—alerting me to anything from tiny typos to blatant blunders—in various draft versions of this book: Nils Berglund, Alan Champneys, Hayato Chiba, Mike Cortez, Peter De Maesschalck, John Guckenheimer, Pavel Gurevich, Annalisa Iuorio, Mike Jeffrey, Hans Kaper, Daniel Karrasch, Chris Jones, Ilona Kosiuk, Steven Lade, Gabriel Lord, Anatoly Neishtadt, Clare Perryman, Sofia Piltz, Nikola Popovic, Jens Rademacher, Martin Rasmussen, Martin Riedler, Stephen Schecter, Jan Sieber, Eric Siero, Peter Szmolyan, Frits Veerman, Martin Wechselberger, and Antonios Zagaris.

Furthermore, I would like to thank the production staff at Springer for the handling of my manuscript. In particular, Achi Dosanjh has been extremely important and tremendously helpful in leading the entire editorial process. Regarding the formatting of the book, I would also like to thank Yuri Kuznetsov, who shared his LaTeX book formatting preamble with me, from which I took some inspiration for the format of this book. Several anonymous referees also provided very valuable feedback, which helped to improve the book.

During the writing of this book I have also benefited from the generous hospitality and financial support of various institutions, including Cornell University, the Max Planck Institute for Physics of Complex Systems, and the Vienna University of Technology. Furthermore, I would like to thank the Austrian Academy of Sciences for support via the "Austrian Programme for Advanced Research and Technology" and the European Commission for support via a "Marie Curie International Reintegration Grant." The final push of this project has been supported through the program "Oberwolfach Leibniz Fellows" by the Mathematisches Forschungsinstitut Oberwolfach.

Although it is obvious for an overview book on a topic, let me stress that I do not make any claims to novelty of its content. I have tried to summarize and condense the extensive literature on multiple time scale dynamics into a more accessible expository format. However, I can certainly say that during the writing of this book, several very natural new ideas arose. I hope that the research-oriented reader will have a similar experience and that this book will provide a starting point for new ideas in multiscale dynamics.

Vienna, Austria Christian Kuehn
2014

Contents

Preface **v**

1 Introduction **1**

1.1 User's Guide . . . 1
1.2 Fast–Slow Systems . . . 8
1.3 The van der Pol Equation . . . 9
1.4 The FitzHugh–Nagumo Equation . . . 10
1.5 Terminology: Fast–Slow Systems . . . 11
1.6 Terminology: Asymptotic Analysis . . . 14
1.7 References . . . 16

2 General Fenichel Theory **19**

2.1 Invariant Manifolds . . . 20
2.2 Perturbations of Invariant Manifolds . . . 28
2.3 Normal Hyperbolicity . . . 40
2.4 Specialization to Fast–Slow Systems . . . 47
2.5 References . . . 50

3 Geometric Singular Perturbation Theory **53**

3.1 Fenichel's Theorem . . . 53
3.2 The Slow Flow . . . 58
3.3 Singularities . . . 61
3.4 Examples . . . 65
3.5 References . . . 70

4 Normal Forms **71**

4.1 The Normally Hyperbolic Case . . . 71
4.2 Fold Points . . . 77
4.3 Fold Curves . . . 80
4.4 Systems of First Approximation . . . 82
4.5 A Note on Linear Systems . . . 85
4.6 References . . . 88

5 Direct Asymptotic Methods **91**
5.1 Elementary Results . . . 91
5.2 Basics of Relaxation Oscillations . . . 95
5.3 Normally Hyperbolic Planar Systems . . . 98
5.4 Generic Folds in Planar Systems . . . 103
5.5 The Period of Relaxation Oscillations . . . 108
5.6 References . . . 111

6 Tracking Invariant Manifolds **113**
6.1 Simple Jumps and Transversality . . . 114
6.2 The FitzHugh–Nagumo Equation . . . 120
6.3 The C^0 Exchange Lemma . . . 124
6.4 The C^1 Exchange Lemma . . . 129
6.5 Fast Waves in the FitzHugh–Nagumo Equation . . . 141
6.6 Strong Exchange Lemmas . . . 145
6.7 BVPs and Periodic Orbits . . . 149
6.8 References . . . 155

7 The Blowup Method **159**
7.1 Basics . . . 160
7.2 Local Computation . . . 167
7.3 The Quasihomogeneous Case . . . 171
7.4 Blowup Analysis of the Generic Fold . . . 178
7.5 Relaxation Oscillations in $\mathbb{R}^2$ . . . 187
7.6 Relaxation Oscillations in $\mathbb{R}^3$ . . . 190
7.7 Remarks on Rescaling . . . 194
7.8 References . . . 196

8 Singularities and Canards **197**
8.1 Folded Singularities in Planar Systems . . . 198
8.2 Singular Hopf Bifurcation in $\mathbb{R}^2$ . . . 205
8.3 The First Lyapunov Coefficient . . . 208
8.4 Canard Explosion . . . 210
8.5 Canards in $\mathbb{R}^3$ . . . 213
8.6 Secondary Canards at Folded Nodes . . . 223
8.7 Singularities beyond Folds . . . 228
8.8 Curvature . . . 233
8.9 References . . . 235

9 Advanced Asymptotic Methods **239**
9.1 Matched Asymptotic Expansions . . . 240
9.2 Further Concepts and Terminology . . . 251
9.3 The Boundary Function Method . . . 255
9.4 WKB Theory . . . 259
9.5 Asymptotics and Blowup . . . 262
9.6 Averaging . . . 265

9.7 Gevrey Asymptotics . . . 268
9.8 Poincaré–Lindstedt and Two-Timing . . . 274
9.9 The Renormalization Group I . . . 279
9.10 The Renormalization Group II . . . 285
9.11 References . . . 290

10 Numerical Methods 295
10.1 Stiff Equations . . . 295
10.2 A-Stability . . . 299
10.3 Boundary Value Problems (BVPs) . . . 303
10.4 Two Standard BVP Methods . . . 306
10.5 Conditioning . . . 308
10.6 Continuation . . . 312
10.7 Heterogeneous Multiscale Methods . . . 315
10.8 Error Analysis for HMM . . . 319
10.9 References . . . 322

11 Computing Manifolds 327
11.1 Basic Techniques . . . 327
11.2 The CSP Method . . . 331
11.3 The ZDP Method . . . 342
11.4 Other Reduction Methods . . . 344
11.5 Saddle-Type Slow Manifolds . . . 348
11.6 Manifolds and Singularities . . . 353
11.7 References . . . 356

12 Scaling and Delay 359
12.1 The Fold Revisited . . . 359
12.2 Delayed Hopf Bifurcation . . . 361
12.3 Delayed Bifurcation and WKB . . . 370
12.4 The Newton Polygon . . . 373
12.5 Power Transformations . . . 376
12.6 Parameterizing Critical Manifolds . . . 379
12.7 Slowly Time-Dependent Systems . . . 384
12.8 Other Newton Polygon Applications . . . 392
12.9 References . . . 395

13 Oscillations 397
13.1 Overview . . . 398
13.2 Folded Nodes . . . 400
13.3 Singular Hopf and Hyperbolic Equilibria . . . 403
13.4 Tourbillon . . . 407
13.5 The Koper Model . . . 408
13.6 Square-Wave / Fold-Homoclinic Bursting . . . 413
13.7 Elliptic / subHopf–Foldcycle Bursting . . . 416
13.8 Three Time-Scale Systems . . . 417
13.9 References . . . 425

14 Chaos in Fast-Slow Systems **431**
14.1 A Basic Idea 432
14.2 Chaotic Dynamics 434
14.3 Smale Horseshoes 436
14.4 Chaos in van der Pol's Equation I 442
14.5 Chaos in van der Pol's Equation II 448
14.6 Hénon-Type Maps 455
14.7 Chaotic Forced Oscillations 460
14.8 Constructing Chaotic Attractors 468
14.9 Stochastic Replacement 471
14.10 References 473

15 Stochastic Systems **477**
15.1 Motivation and Viewpoints 478
15.2 Fenichel Theory for SDEs 482
15.3 Noisy Fold with Slow Drift 488
15.4 Direct Asymptotics 493
15.5 Stochastic Reduction 497
15.6 Random Dynamical Systems 505
15.7 Markov Chains 510
15.8 Large Deviations 512
15.9 Dynkin's Equation and WKB 517
15.10 References 520

16 Topological Methods **525**
16.1 Decomposition of Invariant Sets 525
16.2 Conley Index 530
16.3 Connecting Orbits 535
16.4 Singular Index Pairs 540
16.5 Further Techniques 546
16.6 References 550

17 Spatial Dynamics **553**
17.1 Stability of Traveling Waves I 553
17.2 Stability of Traveling Waves II 558
17.3 Conservation Laws 562
17.4 Lattice Dynamical Systems 573
17.5 Adaptive Neural Fields 578
17.6 References 580

18 Infinite Dimensions **583**
18.1 Semigroups 584
18.2 Invariant Manifolds 587
18.3 Homogenization 595
18.4 Delay Equations with Small Delay 600
18.5 Differential Inclusions 605

18.6 Amplitude Equations . . . 611
18.7 References . . . 615

19 Other Topics 619
19.1 Differential-Algebraic Equations I . . . 620
19.2 Differential-Algebraic Equations II . . . 624
19.3 Nonsmooth Dynamical Systems . . . 627
19.4 Hysteresis . . . 633
19.5 Nonstandard Analysis: Introduction . . . 637
19.6 Nonstandard Analysis: An Application . . . 642
19.7 Averaging and Adiabatic Invariants . . . 646
19.8 Singular Bifurcation Diagrams . . . 649
19.9 Critical Transitions . . . 654
19.10 References . . . 661

20 Applications 665
20.1 Engineering . . . 665
20.2 Neuroscience . . . 668
20.3 Chemical Oscillations . . . 673
20.4 Lasers . . . 676
20.5 Ecology . . . 679
20.6 Pattern Formation . . . 681
20.7 Celestial Mechanics . . . 685
20.8 Systems Biology . . . 688
20.9 Fluid Dynamics . . . 691
20.10 Quantum Mechanics . . . 694
20.11 Networks . . . 696
20.12 References . . . 700

Bibliography 705

Index 799